Effectiveness of the operator splitting for solving the atmospherical shallow water equations

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Abstract *A semi-implicit semi-Lagrangian mixed finite-difference finite-volume model for the shallow water equations on a rotating sphere is considered. The main features of the model are the finite-volume approach for the continuity equation and the vectorial treatment of the momentum equation. Pressure and Coriolis terms in the momentum equation and velocity in the continuity equation are treated semi-implicitly. Discretization of this model led to the introducion, in a previous paper, of a splitting technique which highly reduces the computational effort for the numerical solution. In this paper we solve the full set of equations, without splitting, introducing an* ad hoc *algorithm. A von Neumann stability analysis of this scheme is performed to establish the unconditional stability of the new proposed method. Finally, we compare the efficiency of the two approaches by numerical experiments on a standard test problem. Results show that, due to the devised algorithm, the solution of the full system of equations is much more accurate while slightly increasing the computational cost.*

Introduction

Semi-Lagrangian approximations have been adopted as the basis of several high resolution operational numerical weather prediction models (Ritchie and Beaudoin, 1994; Ritchie *et al.*, 1995). Their principal advantage is the potential for long time steps. Staniforth and Côté (1991) provide an exhaustive review of the early development of semi-Lagrangian methods.

Several other advantages of the Semi-Lagrangian approach have been identified, such as two-time-level schemes in gridpoint models (McDonald and Bates, 1989) and linear grids for spectral models (Williamson, 1997).

Recently, Bartello and Thomas (1996) discussed the cost-effectiveness of semi-Lagrangian advection schemes. Their conclusions are completely consistent with the fact that these schemes present an enormous time-step advantage in large scale models with quasi-geostrophic dynamics; they may still be advantageous for stratospheric models, whereas there is reason to believe that below the synoptic scale in the troposphere their cost-effectiveness is severely reduced.

The split-operator approach is a popular solution algorithm in CFD: operator-splitting techniques have been widely utilized, for example, in atmospheric modeling studies (Hundsdorfer, 1996) to decouple reaction from convection and diffusion, or convection from diffusion. For the incompressible

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Navier-Stokes equations, the method decomposes the momentum equation to solve the linked pressure-velocity problem (Rosenfeld *et al.*, 1991). Though conventional shallow water flow equations have no such problem, in atmospherical shallow water equations there is a similar link between the Coriolis and pressure gradient terms.

We present a numerical model of the inviscid shallow water equations for atmospheric circulation and discuss the effectiveness of the split-operator approach for this problem, comparing the "split" and the "full" versions of the solution algorithm.

The proposed model possesses some specific features, not yet considered in the meteorological literature:

- discretization of the momentum equation in vector form to face the pole singularity arising from the spherical geometry (as proposed by Bates *et al.*, 1990);
- semi-implicit treatment of the Coriolis and pressure terms in the momentum equation and of the divergence terms in the continuity equation to obtain unconditional stability;
- a finite volume approach for the continuity equation to obtain conservation of the geopotential height;
- semi-Lagrangian treatment of advection, with a substepping procedure as in Casulli (1990) to allow long time steps while retaining accuracy;
- solution of the full system of equation using *ad hoc* algorithms, like restarted preconditioned GMRES and Generalized Conjugate Gradients, allows us to achieve a good accuracy at a low cost.

The model

Let us consider the inviscid shallow water equations in spherical components

$$\begin{cases} \frac{d\mathbf{V}}{dt} & = & -f\mathbf{k} \times \mathbf{V} - \nabla_h \Phi \\ \frac{\partial \Phi'}{\partial t} & + & \nabla_h \cdot (\Phi' \mathbf{V}) = 0 \end{cases} \tag{1}$$

Here we use curvilinear coordinates x and y, with $dx = R\cos\varphi d\lambda$, $dy = Rd\varphi$; λ, φ are longitude and latitude, respectively, and R is Earth radius; $\mathbf{V} \equiv (u, v)$ is the wind field with curvilinear components toward the east and the north, respectively; f is the Coriolis parameter, ∇_h is the horizontal gradient operator:

$$\nabla_h = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \left(\frac{1}{R\cos\varphi}\frac{\partial}{\partial \lambda}, \frac{1}{R}\frac{\partial}{\partial \varphi}\right).$$

Finally, $\Phi'(x, y, t)$ is a time-dependent perturbation of the geopotential height field $\Phi(x, y, t)$ about a mean value $\Phi^s(x, y)$: $\Phi' = \Phi - \Phi^s$.

The "full" numerical scheme

To discretize system (1) we use a uniform grid in (λ, φ) variables. The wind components and the geopotential height are staggered as in the Arakawa-C grid: for each cell $D_{i,j}$ centered on (λ_i, φ_j) the geopotential Φ is located at the centre (indexes i,j), the u component at the midpoint of the left and right boundaries (indexes $i \pm 1/2, j$) and the v component at the mid point of the top and bottom boundaries (indexes $i, j \pm 1/2$).

For the continuity equation we adopt a finite-volume discretization, which ensures the conservation of the total geopotential height (as proved in Carfora (2000), generally missed in semi-Lagrangian schemes. In this discretization, the transient term is obtained by first-order difference in time and assuming Φ is constant over the cell; in the divergence terms, the height field will be taken explicitly, whereas the velocity components will be treated semi-implicitly to obtain the unconditional stability of the method, as will be shown in the section headed "Stability results". Then we have the following approximation:

$$
\begin{aligned}
&A_j\left(\Phi^{n+1}_{i,j} - \Phi^n_{i,j}\right)/\Delta t + \\
&\Delta y(\Phi - \Phi^s)^n_{i+\frac{1}{2},j}\left(\theta u^{n+1}_{i+\frac{1}{2},j} + (1-\theta)u^n_{i+\frac{1}{2},j}\right) - \\
&\Delta y(\Phi - \Phi^s)^n_{i-1/2,j}\left(\theta u^{n+1}_{i-1/2,j} + (1-\theta)u^n_{i-1/2,j}\right) + \\
&\Delta x_{j+1/2}(\Phi - \Phi^s)^n_{i,j+1/2}\left(\theta v^{n+1}_{i,j+1/2} + (1-\theta)v^n_{i,j+1/2}\right) - \\
&\Delta x_{j-1/2}(\Phi - \Phi^s)^n_{i,j-1/2}\left(\theta v^{n+1}_{i,j-1/2} + (1-\theta)v^n_{i,j-1/2}\right) = 0
\end{aligned}
\tag{2}
$$

where we indicated with A_j the area of the cell $D_{i,j}$; the implicitness parameter θ is in $[0, 1]$.

This approximation still holds for triangular cells, where the value of the variable v at only one location is involved.

For the momentum equation we use a semi-Lagrangian approach starting from its vectorial formulation and again choose to treat implicitly the Coriolis and pressure terms to obtain unconditional stability of the numerical scheme.

Although not common in meteorological literature, the choice of using a Lagrangian discretization for the momentum equation and a Eulerian one for the continuity equation, first introduced by Casulli (1990), is the basis for several models of tidal circulation (see, for example, Cheng *et al.*, 1993) and its use is consolidated in hydrodynamical literature.

The discrete momentum equations we obtain are:

$$
\begin{pmatrix} 1 & -f\theta\Delta t \\ f\theta\Delta t & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}^{n+1} = -\theta\Delta t\begin{pmatrix} \Phi_x \\ \Phi_y \end{pmatrix}^{n+1} + \begin{pmatrix} \mathcal{L}^u \\ \mathcal{L}^v \end{pmatrix}
\tag{3}
$$

where the first scalar equation is collocated on the u-grid and the second one on the v-grid.

The "Lagrangian terms", the terms to be evaluated at departure points of the characteristic lines, are given by:

$$\begin{pmatrix} \mathcal{L}^u \\ \mathcal{L}^v \end{pmatrix} = \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix} \begin{pmatrix} u + f(1-\theta)v\Delta t - (1-\theta)\Delta t \Phi_x \\ v - f(1-\theta)u\Delta t - (1-\theta)\Delta t \Phi_y \end{pmatrix}_*^n. \quad (4)$$

The coefficients $r_{i,j}$, obtained by geometric considerations, relate the coordinates of arrival and departure points of the trajectories, which we indicate with (λ, φ) and (λ_*, φ_*), respectively. For completeness we recall here the expression for these coefficients:

$$\begin{aligned} r_{1,1} &= \cos\delta\lambda \\ r_{1,2} &= -\sin\delta\lambda \sin\varphi \\ r_{2,1} &= \sin\delta\lambda\sin\varphi_* \\ r_{2,2} &= \cos\delta\lambda\sin\varphi\sin\varphi_* + \cos\varphi\cos\varphi_* \end{aligned}$$

Formal substitution of u^{n+1} and v^{n+1} in the approximated continuity equation leads to a nine-point scheme for Φ^{n+1}:

$$\begin{aligned} &\Big[A_j + \theta^2\Delta t^2\Big(C_{i+\frac{1}{2},j} + C_{i-\frac{1}{2},j} + D_{i,j+\frac{1}{2}} + D_{i,j-\frac{1}{2}}\Big)\Big]\mathbf{\Phi}^{n+1}_{i,j} \\ &-\theta^2\Delta t^2\Big(C_{i+\frac{1}{2},j} - \theta\Delta t(E_{i,j+\frac{1}{2},j} - E_{i,j-\frac{1}{2}})\Big)\mathbf{\Phi}^{n+1}_{i+1,j} \\ &-\theta^2\Delta t^2\Big(C_{i-\frac{1}{2},j} + \theta\Delta t(E_{i+\frac{1}{2},j} - E_{i,j-\frac{1}{2}})\Big)\mathbf{\Phi}^{n+1}_{i-1,j} \\ &-\theta^2\Delta t^2\Big(D_{i,j+\frac{1}{2}} + \theta\Delta t(E_{i+\frac{1}{2},j} - E_{i-\frac{1}{2},j})\Big)\mathbf{\Phi}^{n+1}_{i,j+1} \\ &-\theta^2\Delta t^2\Big(D_{i,j-\frac{1}{2}} - \theta\Delta t(E_{i+\frac{1}{2},j} - E_{i-\frac{1}{2},j})\Big)\mathbf{\Phi}^{n+1}_{i,j-1} \\ &-\theta^3\Delta t^3(E_{i+\frac{1}{2},j} - E_{i+\frac{1}{2}})\mathbf{\Phi}^{n+1}_{i+1,j+1} \\ &-\theta^3\Delta t^3(-E_{i+\frac{1}{2},j} + E_{i,j-\frac{1}{2}})\mathbf{\Phi}^{n+1}_{i+1,j-1} \\ &-\theta^3\Delta t^3(-E_{i-\frac{1}{2},j} + E_{i+\frac{1}{2}})\mathbf{\Phi}^{n+1}_{i-1,j+1} \\ &-\theta^3\Delta t^3(E_{i-\frac{1}{2},j} - E_{i,j-\frac{1}{2}})\mathbf{\Phi}^{n+1}_{i-1,j-1} \end{aligned}$$

$$\begin{aligned} &= A_j\Phi^n_{i,j} - (1-\theta)\Delta t\Big[\Delta y\Big(H_{i+\frac{1}{2},j}u^n_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j}u^n_{i-\frac{1}{2},j}\Big) \\ &+\Delta x_{j+\frac{1}{2}}H_{i,j+\frac{1}{2},}v^n_{i,j+\frac{1}{2}} - \Delta x_{j-\frac{1}{2}}H_{i,j-\frac{1}{2}}v^n_{i,j-\frac{1}{2}}\Big] \\ &-\theta\Delta t\Big[\Delta y\Big(H_{i+\frac{1}{2},j}\mathcal{L}^u_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j}\mathcal{L}^u_{i-\frac{1}{2},j}\Big) \\ &+\Delta x_{j+\frac{1}{2}}H_{i,j+\frac{1}{2}}\mathcal{L}^v_{i,j+\frac{1}{2}} - \Delta x_{j-\frac{1}{2}}H_{i,j-\frac{1}{2}}\mathcal{L}^v_{i,j-\frac{1}{2}}\Big] \end{aligned} \quad (5)$$

where

$$C_{i+\frac{1}{2},j} = \frac{H_{i+\frac{1}{2},j}}{F_j}\frac{\Delta y}{\Delta x_j},$$
$$D_{i,j+\frac{1}{2}} = \frac{H_{i,j+\frac{1}{2}}}{F_{j+1/2}}\frac{\Delta x_{j+1/2}}{\Delta y},$$
$$E_{i,j} = \frac{1}{4}\frac{f_j}{F_j}H_{i,j}.$$

and $H_{i,j} = \Phi^n_{i,j} - \Phi^s_{i,j}, F_j = 1 + f_j^2\theta^2\Delta t^2$.

The "split" numerical scheme

The choice of discretizing implicitly in time Coriolis and pressure terms in the momentum equation leads to the presence of asymmetric extra-diagonal terms in the linear system to be solved. These terms, and the presence of "mixed" derivatives of the geopotential, Φ_y in the u-equation and Φ_x in the v-equation, increase the computational effort for the solution of the system.

For these reasons, in a previous work (Carfora, 2000) we considered a simplifying approach by the introduction of a splitting technique. The basic idea was to split the momentum equation in two parts by considering the total derivative of velocity as the sum of two terms: first (step 1), we take into account only the contribution due to Coriolis terms; then (step 2) the contribution due to pressure terms.

We stress that we introduced two different implicitness parameters (θ and θ_1) for the two steps; this choice has been explained by stability considerations.

Introduction of the splitting technique leads to some simplifications in system (5): in fact, the momentum equation (3) reduces to:

$$\begin{pmatrix} u \\ v \end{pmatrix}^{n+1} + \theta\Delta t \begin{pmatrix} \Phi_x \\ \Phi_y \end{pmatrix}^{n+1} = \begin{pmatrix} \mathcal{L}^u \\ \mathcal{L}^v \end{pmatrix} - (1-\theta)\Delta t \begin{pmatrix} \Phi_x \\ \Phi_y \end{pmatrix}^n_* \tag{6}$$

where now the corresponding expressions of equation (4) for the terms $\mathcal{L}$ contain only Coriolis terms:

$$\begin{pmatrix} \mathcal{L}^u \\ \mathcal{L}^v \end{pmatrix} = \frac{1}{1+f^2\theta_1^2\Delta t^2}\begin{pmatrix} 1 & f\theta_1\Delta t \\ -f\theta_1\Delta t & 1 \end{pmatrix}\begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}\begin{pmatrix} u + f(1-\theta_1)v\Delta t \\ v - f(1-\theta_1)u\Delta t \end{pmatrix}^n_*. \tag{7}$$

It follows that the linear system to be solved for the geopotential reduces to:

$$\left[A_j + \theta^2\Delta t^2\left(C_{i+\frac{1}{2},j} + C_{i-\frac{1}{2},j} + D_{i,j+\frac{1}{2}} + D_{i,j-\frac{1}{2}}\right)\right]\mathbf{\Phi}^{n+1}_{i,j}$$
$$-\left[\theta^2\Delta t^2 C_{i+\frac{1}{2},j}\right]\mathbf{\Phi}^{\mathbf{n+1}}_{\mathbf{i+1,j}} - \left[\theta^2\Delta t^2 C_{i-\frac{1}{2},j}\right]\mathbf{\Phi}^{n+1}_{i-1,j} \tag{8}$$
$$-\left[\theta^2\Delta t^2 D_{i,j+\frac{1}{2}}\right]\mathbf{\Phi}^{\mathbf{n+1}}_{\mathbf{i,j+1}} - \left[\theta^2\Delta t^2 D_{i,j-\frac{1}{2}}\right]\mathbf{\Phi}^{n+1}_{i,j-1}$$

$$= A_j\Phi^n_{i,j} - (1-\theta)\Delta t\left[\Delta y\left(H_{i+\frac{1}{2},j}u^n_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j}u^n_{i-\frac{1}{2},j}\right)\right.$$
$$\left. + \Delta x_{j+\frac{1}{2}}H_{i,j+\frac{1}{2}}v^n_{i,j+\frac{1}{2}} - \Delta x_{j-\frac{1}{2}}H_{i,j-\frac{1}{2}}v^n_{i,j-\frac{1}{2}}\right]$$
$$-\theta\Delta t\left[\Delta y\left(H_{i+\frac{1}{2},j}\mathcal{L}^u_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j}\mathcal{L}^u_{i-\frac{1}{2},j}\right)\right.$$
$$\left. + \Delta x_{j+\frac{1}{2}}H_{i,j+\frac{1}{2}}\mathcal{L}^v_{i,j+\frac{1}{2}} - \Delta x_{j-\frac{1}{2}}H_{i,j-\frac{1}{2}}\mathcal{L}^v_{i,j-\frac{1}{2}}\right]$$

where the same notations of system (5) have been used, except for the terms $\mathcal{L}$, defined by equation (7).

The semi-Lagrangian terms

To solve any of the two numerical schemes presented (3-5) and (6-8), the evaluation of the "semi-Lagrangian terms" $\mathcal{L}$ in equation (4) or in equation (7), respectively, requires the velocity components u^n_* and v^n_* at the departure points of the characteristic lines.

To determine these departure points we integrate (backward in time) the following system of ordinary differential equations (the "characteristic system") from time t^{n+1} to t^n at any vertex of the u grid and v grid:

$$\begin{cases} R\frac{d}{dt}\lambda\cos\varphi & = u(\lambda,\varphi) \\ R\frac{d\varphi}{dt} & = v(\lambda,\varphi). \end{cases} \tag{9}$$

In order to reconstruct more accurately the characteristic lines we use a sub-stepping procedure first introduced by Casulli (1990) and solve the system of ordinary differential equations evaluating λ, θ in N intermediate time steps τ_k, where N is chosen with a global condition on the Courant numbers such that in any substep

$$\max\left[\frac{|u|\Delta t}{N\Delta x}, \frac{|v|\Delta t}{N\Delta y}\right] \leq 1.$$

Two different numerical schemes have been tested: the first one is the explicit Euler method, which is first-order accurate:

$$
\begin{cases}
\lambda^{(k-1)} & = & \lambda^{(k)} - \tau \dfrac{u^{(k)}}{R\cos\varphi^{(k)}} \\
\varphi^{(k-1)} & = & \varphi^{(k)} - \tau \dfrac{v^{(k)}}{R}
\end{cases}
\tag{10}
$$

The second method is the (Runge-Kutta) Heun method, which is second-order accurate:

$$
\begin{cases}
\lambda^{(k-\frac{1}{2})} & = & \lambda^{(k)} - \tau \dfrac{u^{(k)}}{R\cos\varphi^{(k)}} \\
\varphi^{(k-\frac{1}{2})} & = & \varphi^{(k)} - \tau \dfrac{v^{(k)}}{R} \\
\lambda^{(k-1)} & = & \lambda^{(k)} - \dfrac{\tau}{2}\left[\dfrac{u^{(k-\frac{1}{2})}}{R\cos\varphi^{(k-\frac{1}{2})}} + \dfrac{u^{(k)}}{R\cos\varphi^{(k)}}\right] \\
\varphi^{(k-1)} & = & \varphi^{(k)} - \dfrac{\tau}{2}\left[\dfrac{v^{(k-\frac{1}{2})}}{R} + \dfrac{v^{(k)}}{R}\right]
\end{cases}
\tag{11}
$$

Then, values of u and v at the foot of characteristic lines are determined by bicubic Lagrange interpolation. Particular attention has been paid to the interpolation procedure near the poles: as described in detail in Amato and Carfora (2000), for the "border cells" a suitable use of the variables in grid points across the pole allows us to retain the same accuracy of the "internal cells".

Stability results

In this section we prove the unconditional stability of the linearized system of equations (2) and (3) in the case where the implicitness parameter θ is in $[0.5, 1]$. The proofs of the following theorems are quite technical; however, we report them here for completeness.

To obtain from equations (2), (3) and (4) the linearized discrete equations we do some simplifications: if we indicate with f a constant value for the Coriolis parameter and with $\overline{H}$ the mean geopotential height, we can introduce the constants $K_j = \sqrt{\overline{H}}/\Delta x_j$ and $K = \sqrt{\overline{H}}/\Delta y$; also we introduce the new variable $\Psi = \Phi/\sqrt{\overline{H}}$; moreover, we assume that $A_j = \Delta x_j \Delta y$ and that $\Delta x_{j+\frac{1}{2}} = \Delta x_j = \Delta x_{j-\frac{1}{2}}$.

With these positions, we obtain the system:

$$
\begin{cases}
u^{n+1}_{i+\frac{1}{2},j} - f\theta\Delta t v^{n+1}_{i+\frac{1}{2},j} & +\theta K_j \Delta t\left(\Psi^{n+1}_{i+1,j} - \Psi^{n+1}_{i,j}\right) = \mathcal{L}^u_{i+\frac{1}{2},j} \\
f\theta\Delta t u^{n+1}_{i,j+\frac{1}{2}} + v^{n+1}_{i,j+\frac{1}{2}} & +\theta K\Delta t\left(\Psi^{n+1}_{i,j+1} - \Psi^{n+1}_{i,j}\right) = \mathcal{L}^v_{i+\frac{1}{2}} \\
\Psi^{n+1}_{i,j} + \theta K_j \Delta t\left(u^{n+1}_{i+\frac{1}{2},j} - u^{n+1}_{i-\frac{1}{2},j}\right) & +\theta K\Delta t\left(v^{n+1}_{i,j+\frac{1}{2}} - v^{n+1}_{i,j-\frac{1}{2}}\right) \\
= \Psi^n_{i,j} - (1-\theta)K_j\Delta t\left(u^n_{i+\frac{1}{2},j} - u^n_{i-\frac{1}{2},j}\right) & -(1-\theta)K\Delta t\left(v^n_{i,j+\frac{1}{2}} - v^n_{i,j-\frac{1}{2}}\right)
\end{cases}
\tag{12}
$$

Then we introduce a Fourier mode for the dependent variables u, v and Ψ and carry out a stability analysis on the corresponding amplitude functions. We write the equations for a single mode $\overline{w}^n e^{Iix} e^{Ijy}$, where $\overline{w}^n$ is the amplitude of the variable w (w standing for u, v, Ψ) at the time level n:

$$B\overline{\mathbf{w}}^{n+1} = RC\overline{\mathbf{w}}^n$$

where at the time level k

$$\overline{\mathbf{w}}^k = \left(\overline{u}^k, \overline{v}^k, \overline{\Psi}^k\right);$$

$$B = \begin{pmatrix} 1 & -f\theta\Delta t & 2I\theta\Delta t K_j \sin(\frac{x}{2}) \\ f\theta\Delta t & 1 & 2I\theta\Delta t K \sin(\frac{y}{2}) \\ 2I\theta\Delta t K_j \sin(\frac{x}{2}) & 2I\theta\Delta t K \sin(\frac{y}{2}) & 1 \end{pmatrix}$$

$$R = E\begin{pmatrix} r_{1,1} & r_{1,2} & 0 \\ r_{2,1} & r_{2,2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where E is the amplification factor of the interpolation procedure (we suppose $|E| = 1$) and

$$C = \begin{pmatrix} 1 & f(1-\theta)\Delta t & -2I(1-\theta)\Delta t K_j \sin(\frac{x}{2}) \\ -f(1-\theta)\Delta t & 1 & -2I(1-\theta)\Delta t K \sin(\frac{y}{2}) \\ -2I(1-\theta)\Delta t K_j \sin(\frac{x}{2}) & -2I(1-\theta)\Delta t K \sin(\frac{y}{2}) & 1 \end{pmatrix}.$$

To obtain a necessary condition on the stability of the numerical method we have to show that the spectral radius of the amplification matrix $B^{-1}RC$ is not greater than 1.

To do that, we need a preliminary result.

Lemma 1 *It is*

$$\|B^{-1}C\|_2 = \left(1 + (1-2\theta)\frac{A}{1+\theta^2 A}\right)^{1/2}$$

where $A = (f\theta\Delta t)^2 + 4(K_j^2 \sin^2(x/2) + K^2 \sin^2(y/2))$.

Proof. Since $B^{-1}C$ is a normal matrix (a matrix M is said to be normal iff it commutes with its conjugate transpose), its L_2 norm coincides with its spectral radius. Then it suffices to evaluate the eigenvalues of $B^{-1}C$. They are

$$\lambda_1 = 1; \qquad \lambda_{2,3} = \frac{1-\theta(1-\theta)A \pm I\sqrt{A}}{1+\theta^2 A}.$$

Then,

$$|\lambda_{2,3}| = \sqrt{\lambda_2\lambda_3} = \left(1+(1-2\theta)\frac{A}{1+\theta^2 A}\right)^{1/2}$$

and, since this last quantity is not less than $\lambda_1 = 1$, this is exactly the spectral radius of $B^{-1}C$.

Theorem 1 *The spectral radius of the amplification matrix of the linearized system (equation12) is not greater than 1 iff the implicitness parameter θ is in* $[0.5, 1]$.

Proof. From the definition of eigenvalues it is self-evident that $B^{-1}RC$ and RCB^{-1} have the same eigenvalues, and then the same spectral radius. Moreover, it is well-known that, for any matrix M, $\rho(M) \leq \|A\|$. Finally, since R is a submatrix of an orthogonal (rotation) matrix, it is $\|R\| \leq 1$. These three considerations lead us to the desired result: since it is

$$\rho(B^{-1}RC) = \rho(RCB^{-1}) \leq \|RCB^{-1}\|_2 \leq \|R\|_2\|CB^{-1}\|_2$$

by application of the previous lemma we have

$$\rho(B^{-1}RC) \leq \left(1+(1-2\theta)\frac{A}{1+\theta^2 A}\right)^{1/2}$$

and this quantity is not greater than 1, provided that $1-2\theta \leq 0$, i.e. $\theta \geq 0.5$.

In Carfora (2000) a corresponding result for the unconditional stability of the split numerical method was proven, with a suitable choice of the implicitness parameters θ and θ_1; moreover, a sharp estimate of the instability due to Coriolis terms was obtained. Indeed, the following theorem holds:

Theorem 2 *It is*

$$1 \leq \|B^{-1}C\|_2 \leq \max\left(1, \sqrt{\frac{1+f^2(1-\theta_1)^2\Delta t^2}{1+f^2\theta_1^2\Delta t^2}}\right) + 2\left[\overline{H}\left(\frac{1}{\Delta x_1^2}+\frac{1}{\Delta y^2}\right)\right]^{1/2}(1-\theta)\Delta t \quad (13)$$

Then, the considered numerical method is unconditionally stable for $\theta = 1$ provided that $\theta_1 \in [0.5, 1]$. Moreover, for $\theta_1 \in [0, 0.5)$ we have

$$\|B^{-1}C\|_2 \approx 1 + f^2 \Delta t^2. \tag{14}$$

Computational issues

The numerical solution is obtained by solving a linear system for the geopotential height.

For the split scheme the matrix obtained from system (8) is sparse, symmetric and strictly diagonally-dominant with only seven non-zero diagonals, since for periodicity reasons on the Earth's surface the five-points scheme leads to a seven-diagonals structure. The system can be solved by a classical Conjugate Gradients algorithm with a simple diagonal preconditioner, as described in Carfora (2000).

For the full scheme the matrix obtained from system (5) is also sparse and band-structured with only 15 non-zero diagonals.

The author tested on this problem the Generalized Conjugate Gradients algorithm introduced by Concus and Golub (1976) and found that this algorithm performs really well on this kind of "nearly-symmetric" system, where the symmetric part of the involved matrix is dominant over the antisymmetric part: indeed, we found that this algorithm costs approximately three times more than Conjugate Gradients, since it requires about three Conjugate Gradients calls per timestep.

However, this system can be solved even more efficiently by Restarted GMRES (Saad and Schultz, 1986) combined with a suitable preconditioner. The results we show in the following Section confirm the good cost-effectiveness of this algorithm.

Numerical experiments

We performed some numerical experiments to compare the two versions of the numerical method, that we indicated as "full" and "split".

To do this, we solved one of the test problems proposed by Williamson *et al.* (1992). In the last few years, this set had been used for the validation of several shallow water numerical models of the atmosphere.

Our test is a steady state solution to the non-linear shallow water equations, that is a solid body rotation or zonal flow with the corresponding geostrophic height field.

The test comprises an initial height profile (for simplicity, a cosine bell) which rotates with constant angular velocity Ω around the Earth's axis (through the Poles) and we consider this rotation in a spherical coordinate system (λ, φ) having its North Pole at point P (not coinciding with the physical North Pole (NP) in general). If $(0, \varphi_0)$ are NP coordinates in this system, the analytical solution to this test problem is given by:

$$\Phi = \Phi_0 - \left[\Omega R u_0 + \frac{u_0^2}{2}\right](\cos\lambda\cos\varphi\sin\varphi_0 + \sin\varphi\cos\varphi_0)^2 \qquad (15)$$

whereas wind components are

$$\begin{cases} u &= u_0[\cos\varphi\cos\varphi_0 + \cos\lambda\sin\varphi\sin\varphi_0] \\ v &= -u_0\sin\lambda\sin\varphi_0 \end{cases} \qquad (16)$$

and where $u_0 = 2\pi R/12$ days and $\Phi_0 = 2.94 \times 10^4 m^2/s^2$.

We tested our model on a slightly rotated grid (φ_0 = 0.05 rad) and on a fully rotated grid ($\varphi_0 = \pi/2 - 0.05$ rad).

In both cases, executions were made for several values of Δt and for the grid resolution of 180 × 90 grid points.

According to the stability results, in the "full" scheme (equations (3)-(5)) we set $\theta = 0.5$, whereas in the "split" scheme (equations (6)-(8)) we set $\theta = 1$, $\theta_1 = 0.5$.

The global relative errors on the retrieved fields in L_1, L_2 and L_∞ norms (indexes l_1, l_2, l_∞) have been calculated after five days of simulation.

Tables I and II compare the accuracy of the two schemes showing the error indicators for the geopotential and for the wind field after a five-day simulation in the case of a slightly rotated grid ($\varphi_0 = \pi/2$ – 0.05 rad). Tables III and IV make the same comparison for the fully rotated grid (φ_0 = 0.05 rad).

For the "full" scheme, we report here the results obtained with the two considered solution algorithms (Restarted GMRES and Generalized CG).

These tables show that the accuracy of the method, in both cases, does not depend on the rotation of the numerical grid. Indeed, the case of a full-rotated grid, with higher Courant numbers, gives the same error levels of a slightly-rotated grid. In all the considered cases, the full scheme (in its two variants) is much more accurate than the split one. Moreover, the GMRES version is slightly more accurate than the GCG version of the full scheme.

Finally, Table V shows the cost of the schemes in terms of computer time. Tests were performed on an Alpha Server 2100/250. This machine is rated at about 120Mflop/s in the LINPACK benchmark. They confirm that the full scheme costs about three-four times more than the split scheme.

	Split scheme			Full scheme – GMRES			Full scheme – GCG		
Δt	$L_1(\Phi)$	$L_2(\Phi)$	$L_\infty(\Phi)$	$L_1(\Phi)$	$L_2(\Phi)$	$L_\infty(\Phi)$	$L_1(\Phi)$	$L_2(\Phi)$	$L_\infty(\Phi)$
5'	0.272e-1	0.300e-1	0.460e-1	0.112e-3	0.122e-3	0.196e-3	0.115e-3	0.132e-3	0.373e-3
10'	0.505e-1	0.562e-1	0.926e-1	0.256e-3	0.277e-3	0.383e-3	0.290e-3	0.354e-3	0.155e-2
20'	0.880e-1	0.989e-1	0.178	0.662e-3	0.723e-3	0.105e-2	0.102e-2	0.130e-2	0.752e-2
40'	0.137	0.155	0.292	0.255e-2	0.300e-2	0.604e-2	0.452e-2	0.583e-2	0.283e-1

Notes: Grid 180 × 90 – North Pole at $\pi/2$ – 0.5 rad.; T max = 120 hours

Table I. Comparison of the error indicators for the geopotential in a slightly rotated grid

However, it must be stressed that, in this series of tests, all the schemes use a simple diagonal preconditioner. It has to be expected that, with a "tailored" implementation of the GMRES solution algorithm in the full case and a suitable choice for the preconditioner, the cost of this last scheme will decrease, leading to a far more competitive algorithm.

Table II.
Comparison of the error indicators for the velocity in a slightly rotated grid

	Split scheme			Full scheme – GMRES			Full scheme – GCG		
Δt	$L_1(v)$	$L_2(v)$	$L_\infty(v)$	$L_1(v)$	$L_2(v)$	$L_\infty(v)$	$L_1(v)$	$L_2(v)$	$L_\infty(v)$
5'	0.872e-1	0.988e-1	0.125	0.142e-2	0.169e-2	0.244e-2	0.145e-2	0.171e-2	0.378e-2
10'	0.165	0.184	0.227	0.266e-2	0.328e-2	0.472e-2	0.291e-2	0.348e-2	0.902e-2
20'	0.296	0.323	0.382	0.541e-2	0.647e-2	0.936e-2	0.701e-2	0.836e-2	0.412e-1
40'	0.481	0.512	0.579	0.135e-1	0.140e-1	0.184e-1	0.229e-1	0.271e-1	0.126

Notes: Grid 180 × 90 – North Pole at 0.5 rad.; T max = 120 hours

Table III.
Comparison of the error indicators for the geopotential in a fully rotated grid

	Splitted scheme			Full scheme – GMRES			Full scheme – GCG		
Δt	$L_1(\Phi)$	$L_2(\Phi)$	$L_\infty(\Phi)$	$L_1(\Phi)$	$L_2(\Phi)$	$L_\infty(\Phi)$	$L_1(\Phi)$	$L_2(\Phi)$	$L_\infty(\Phi)$
5'	0.271e-1	0.299e-1	0.459e-1	0.651e-4	0.786e-4	0.158e-3	0.722e-4	0.890e-4	0.207e-3
10'	0.505e-1	0.561e-1	0.926e-1	0.150e-3	0.166e-3	0.320e-3	0.231e-3	0.281e-3	0.689e-3
20'	0.880e-1	0.988e-1	0.178	0.463e-3	0.529e-3	0.827e-3	0.904e-3	0.110e-2	0.264e-2
40'	0.137	0.155	0.292	0.242e-2	0.305e-2	0.651e-2	0.409e-2	0.506e-2	0.116e-1

Notes: Grid 180 × 90 – North Pole at $\pi/2 - 0.5$ rad.; T max = 120 hours

Table IV.
Comparison of the error indicators for the velocity in a fully rotated grid

	Splitted scheme			Full scheme – GMRES			Full scheme – GCG		
Δt	$L_1(v)$	$L_2(v)$	$L_\infty(v)$	$L_1(v)$	$L_2(v)$	$L_\infty(v)$	$L_1(v)$	$L_2(v)$	$L_\infty(v)$
5'	0.868e-1	0.985e-1	0.126	0.107e-2	0.126e-2	0.783e-2	0.112e-2	0.132e-2	0.822e-2
10'	0.164	0.183	0.227	0.201e-2	0.245e-2	0.521e-2	0.254e-2	0.296e-2	0.784e-2
20'	0.296	0.322	0.382	0.421e-2	0.508e-2	0.138e-1	0.753e-2	0.846e-2	0.227e-1
40'	0.479	0.511	0.579	0.118e-1	0.135e-1	0.393e-1	0.285e-1	0.310e-1	0.727e-1

Notes: Grid 180 × 90 – North Pole at 0.5 rad.; T max = 120 hours

Table V.
Execution times for the three algorithms

Timestep	NP latitude	Time (split)	Time (full-GMRES)	Time (full-GCG)
10'	$\pi/2 - 0.05$	9'	46'	44'
10'	0.05	18'	80'	63'
40'	$\pi/2 - 0.05$	16'	68'	56'
40'	0.05	28'	112'	91'

Conclusions

In this paper, we have presented a gridpoint numerical method for solving the atmospherical shallow water. Its unconditional stability, conservation properties and potential for long timesteps, along with its ease of implementation, make it an attractive prototype for a global circulation model.

We also introduced the split of the shallow water equations for the atmosphere, supposing that, as in other fields of CFD, we could obtain a real improvement in the efficiency of the considered numerical method. After a series of numerical experiments, performed on the standard test set for atmospheric shallow water (Williamson *et al.*, 1992), we have found that, for this particular problem, the splitting technique is not effective, since the price we pay to reduce the computational cost is too high in terms of loss of accuracy; it will be preferable to solve the full system of equations, while considering some other improvements to accelerate the solution of the scheme or to reduce the cost of the evaluation of some terms.

References

Amato, U. and Carfora, M.F. (2000), "Semi-Lagrangian treatment of advection on the sphere with accurate spatial and temporal approximations", *Mathematical and Computer Modelling*, Vol. 32, pp. 981-95.

Bartello, P. and Thomas, S.J. (1996), "The cost-effectiveness of semi-Lagrangian advection", *Mon. Wea. Rev.*, Vol. 124, pp. 2883-97.

Bates, J.R., Semazzi, F.H.M., Higgins, R.W. and Barros, S.R.M. (1990), "Integration of the shallow water equation on the sphere using a vector semi-Lagrangian scheme with a multigrid solver", *Mon. Wea. Rev.*, Vol. 118, pp. 1615-27.

Carfora, M.F. (2000), "An unconditionally stable semi-Lagrangian method for the spherical atmospherical shallow water equations", *Int. J. for Numer. Meth. in Fluids*, Vol. 34 No. 6, pp. 527-58.

Casulli, V. (1990), "Semi-implicit finite difference methods for the two-dimensional shallow water equations", *J. Comp. Phys.*, Vol. 86, pp. 56-74.

Cheng, R.T., Casulli, V. and Gartner, J.W. (1993), "Tidal, residual, intertidal mudflat (TRIM) model and its application to San Francisco Bay, California", *Estuar. Coastal Shelf Sci.*, Vol. 36, pp. 235-80.

Concus, P. and Golub, G.H. (1976), "A generalized conjugate gradient method for non-symmetric systems of linear equations", *Lecture Notes in Economics and Mathematical Systems*, Vol. 134, Springer-Verlag, Berlin-Heidelberg-New York, NY, pp. 56-65.

Hundsdorfer, W. (1996), "Numerical solution of advection-diffusion-reaction equations", Lecture Notes, Report NM-N9603, CWI, Amsterdam; http://info4u.cwi.nl

McDonald, A. and Bates, J.R. (1989), "Semi-Lagrangian integration of a gridpoint shallow water model on the sphere", *Mon. Wea. Rev.*, Vol. 117, pp. 130-37.

Ritchie, H. and Beaudoin, C. (1994), "Approximation and sensitivity experiments with a baroclinic semi-Lagrangian spectral model", *Mon. Wea. Rev.*, Vol. 122, pp. 2391-9.

Ritchie, H., Temperton, C., Simmons, A., Hortal, M., Davies, T., Dent, D. and Harmud, M. (1995), "Implementation of the semi-Lagrangian method in a high-resolution version of the ECMWF forecast model", *Mon. Wea. Rev.*, Vol. 123, pp. 489-514.

Rosenfeld, M, Kwak, D. and Vinokur, M. (1991), "A fractional step solution method for the unsteady incompressible Navier-Stokes equations in generalized co-ordinate systems", *J. Comp. Phys*., Vol. 94, pp. 102-37.

Saad, Y. and Schultz, M. (1986), "GMRES: a generalized minimal residual algorithm for solving non-symmetric linear systems", *SIAM J. Sci. Stat. Comp*., Vol. 7, pp. 856-69.

Staniforth, A. and Côté, J. (1991), "Semi-Lagrangian integration schemes for atmospheric models – a review", *Mon. Wea. Rev*., Vol. 119, pp. 2206-23.

Williamson, D.L. (1997), "Climate simulations with a spectral, semi-Lagrangian model with linear grids", in Lin, C., Laprise, R. and Ritchie, H. (Eds), *Numerical Methods in Atmospheric and Oceanic Modeling. The André J. Robert Memorial Volume*, pp. 279-92.

Williamson, D.L., Drake, J.B., Hack, J.J., Jakob, R. and Swarztrauber, P.N. (1992), "A standard test set for numerical approximations to the shallow water equations in spherical geometry", *J. Comp. Phys*., Vol. 102, pp. 211-24.